Heyting Duality

Vikraman Choudhury

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1 Heyting Duality

Pairs of "equivalent" concepts or phenomena are ubiquitous in mathematics, and *dualities* relate such two different, even opposing concepts. Stone's Representation theorem (1936), due to Marshall Stone, states the duality between Boolean algebras and topological spaces. It shows that, for classical propositional logic, the Lindenbaum-Tarski algebra of a set of propositions is isomorphic to the clopen subsets of the set of its valuations, thereby exposing an algebraic viewpoint on logic.

In this essay, we consider the case for intuitionistic propositional logic, that is, a duality result for Heyting algebras. It borrows heavily from an exposition of Stone and Heyting duality by van Schijndel and Landsman [2017].

1.1 Preliminaries

Definition (Lattice)**.** *A lattice is a poset which admits all finite meets and joins. Categorically, it is a* $(0, 1)$ *– category (or a thin category) with all finite limits and finite colimits. Alternatively, a lattice is an algebraic structure in the signature* $(\wedge, \vee, 0, 1)$ *that satisfies the following axioms.*

- $\wedge$ *and* $\vee$ *are each idempotent, commutative, and associative with respective identities 1 and 0.*
- *the absorption laws,* $x \vee (x \wedge y) = x$*, and* $x \wedge (x \vee y) = x$*.*

Definition (Distributive lattice)**.** *A distributive lattice is a lattice in which* $\wedge$ *and* $\vee$ *distribute over each other, that is, the following distributivity axioms are satisfied. Categorically, this makes it a distributive category.*

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Definition (Complements in a lattice). *A complement of an element x of a lattice is an element y such that, $x \wedge y = 0$ and $x \vee y = 1$. Complements need not be unique. If it is unique, we denote the complement of x by $\neg x$.*

Definition (Boolean algebra). *A Boolean algebra $\mathfrak{B}$ is a distributive lattice with unique complements.*

1.2 Stone Duality

We first state the Stone Representation Theorem, which establishes the duality of Boolean algebras and Stone spaces.

Definition (Stone space). *A topological space (X, τ) is totally disconnected if τ has a basis of clopen sets. A Stone space is a totally disconnected compact Hausdorff space.*

Definition (PF). *For $\mathfrak{L}$ a bounded distributive lattice, let $PF(\mathfrak{L})$ be the set of prime filters. For $a \in \mathfrak{L}$, we define*

$$\begin{aligned} F &: \mathfrak{L} \to PF(\mathfrak{L}) \\ a &\mapsto F_a := \{ P \in PF(\mathfrak{L}) \mid a \in P \} \end{aligned}$$

Note that $PF(\mathfrak{L})$ is a also a poset in its own right, with respect to inclusion. It is easy to check that it is also a bounded distributive lattice.

Definition (Cl). *For X a topological space, let $Cl(X)$ be the set of clopen sets of X.*

Definition (CU). *For X a topological space, let $CU(X)$ the set of clopen up-sets of X, ordered by inclusion.*

Lemma. *The map $F : \mathfrak{L} \to PF(\mathfrak{L})$ is a homomorphism. It preserves joins, meets, 0, 1, and complements.*

Proof. $F_0 = \varnothing$ because no prime filter contains 0. $F_1 = PF(\mathfrak{L})$ because every prime filter contains 1. $F_a \cup F_b \subseteq F_{a \vee b}$ because filters are up-sets and $a \in P$ or $b \in P$ implies $a \vee b \in P$. Moreover, $F_{a \vee b} \subseteq F_a \cup F_b$ because $P \in F_{a \vee b}$ is prime, hence $a \vee b \in P$ implies $a \in P$ or $b \in P$. The case for meets follows dually because filters are closed under finite meets, and are up-sets.

If $\mathfrak{L}$ has complements, we want to prove that $\neg F_a = F'_a = F_{\neg a}$. Since $P \in PF(\mathfrak{L})$ is proper, it cannot contain both a and $\neg a$. Since P is prime, and hence an ultrafilter (using the ultrafilter lemma), it contains either a or $\neg a$. This means that the set of prime filters containing $\neg a$ is precisely the set of prime filters not containing a. □

Definition (Topology τ on $PF(\mathfrak{L})$)**.** *We equip $PF(\mathfrak{L})$ with a topology τ as follows. For $a, b \in \mathfrak{L}$, let $S := \{F_a \mid a \in \mathfrak{L}\} \cup \{F'_b \mid b \in \mathfrak{L}\}$ be a subbasis. Then $T := \{F_a \cap F'_b \mid a, b \in \mathfrak{L}\}$ is a basis of τ.*

Lemma. *If $\mathfrak{L}$ is a distributive lattice, then $PF(\mathfrak{L})$ is a Stone space.*

Proof. Let A be in T, the basis of τ, there are $a, b \in \mathfrak{L}$ such that $A = F_a \cap F'_b$. Since A is in the basis of τ, A is open. Now, $A' = (F_a \cap F'_b)' = F'_a \cup F_b$ by de Morgan's laws. Since F'_a and F_b are in the basis, their union A' is open, so A is closed.

Let P, Q be prime filters with $P \neq Q$. Then either $P \nsubseteq Q$ or $Q \nsubseteq P$. If $P \nsubseteq Q$, there is an $a \in \mathfrak{L}$ with $a \in P$ and $a \notin Q$, so $P \in F_a$ and $Q \in F'_a$. Since P and Q are separated by disjoint open sets, $PF(\mathfrak{L})$ is Hausdorff.

To show that $PF(\mathfrak{L})$ is compact, we need to show that any open cover $\mathscr{U}$ of $PF(\mathfrak{L})$ has a finite subcover. If $U \in \mathscr{U}$, then $U = \bigcup_{a,b \in A} \{F_a \cap F'_b\}$, for some $A \subseteq \mathfrak{L}$. Since $\mathscr{U}$ is a cover, for fixed $A_1, A_2 \subseteq A$,

$$PF(\mathfrak{L}) \subseteq \bigcup_{a,b \in A} \{F_a \cap F'_b\} \subseteq \bigcup_{a \in A_1} \{F_a\} \cup \bigcup_{a \in A_2} \{F'_b\}$$

which is also an open cover of $PF(\mathfrak{L})$. Hence, $\bigcap_{b \in A_2} F_b \subseteq \bigcup_{a \in A_1} F_a$.

Let I be the ideal generated by $a \in A_1$ and G the filter generated by $b \in A_2$. A proof by contradiction and the use of the prime ideal theorem shows that $G \cap I \neq \varnothing$. If $x \in G \cap I$, $x \geq b_* := b_1 \wedge \ldots \wedge b_n$ for some $b_i \in A_2$. Also, $x \leq a_* = a_1 \vee \ldots \vee a_m$ for some $a_i \in A_1$. Because F_is are prime filters, $F_{b_1} \cap \ldots \cap F_{b_n} \subseteq F_{b_*} \subseteq F_{a_*} \subseteq F_{a_1} \cup \ldots \cup F_{a_m}$. This gives a covering of $PF(\mathfrak{L})$ which makes it compact. □

Lemma. *If X is a Stone space, then $Cl(X)$ is a Boolean algebra.*

Proof. This follows by a tedious checking of the axioms of a Boolean algebra, using the fact that clopenness is preserved under intersection, union and complement. □

We now claim that PF and Cl establish an equivalence, which proves the Stone representation theorem.

Theorem (Stone Representation Theorem). *The map $f : \mathfrak{B} \to Cl(PF(\mathfrak{B}))$ given by $b \mapsto F_b$ is an isomorphism of Boolean algebras. The map $g : X \to PF(Cl(X))$ given by $x \mapsto F_U := \{ U \in Cl(X) \mid x \in U \}$ is a homeomorphism.*

Proof. First, we notice that $Cl(PF(\mathfrak{B}))$ is a Boolean algebra by the above lemma, hence a subalgebra of the power set of $PF(\mathfrak{B})$. Since F_b is an element of the subbase S of τ, it is open. Since $(F_b)'$ is also an element, F_b is closed, hence clopen.

To show that f is injective, consider $a \neq b$. We want to show that $F_a \neq F_b$. Either $a \nleq b$ or $b \nleq a$. Using $a \nleq b$, let $G =\uparrow a$ be the filter generated by a and $I =\downarrow b$ be the ideal generated by b, so that $G \cap I = \emptyset$. Then there is a prime filter P with $G \subseteq P$ and $P \cap I = \emptyset$. Hence, $a \in P$ and $b \notin P$, so $P \in F_a$ and $P \in F_b$, so that $F_a \neq F_b$.

To show that f is surjective, consider C a clopen subset of $PF(\mathfrak{B})$, we need to show that C is in the image of f. Since C is open, for $B_1, B_2 \subseteq \mathfrak{B}$,

$$C = \bigcup_{b \in B_1, c \in B_2} \{ F_b \cap F_c' \} = \bigcup_{b \in B_1, c \in B_2} \{ P \in PF(\mathfrak{B}) \mid b \in P, c \notin P \}$$

Since P is also an ultrafilter, $c \notin P$ implies $c' \in P$, and since P is closed under meets, $b \wedge c' \in P$. Hence,

$$C = \bigcup_{b \in B_1, c \in B_2} \{ P \in PF(\mathfrak{B}) \mid b \wedge c' \in P \} = \bigcup_{d \in B_3} \{ P \in PF(\mathfrak{B}) \mid d \in P \} = \bigcup_{d \in B_3} F_d$$

Since F_d is open for each d, $\{ F_d \mid d \in B_3 \}$ is an open cover of C. Since C is closed and $PF(\mathfrak{B})$ is compact, C is also compact. Hence, $C = F_{d_1} \cup \ldots \cup F_{d_n} = F_{d_1 \vee \ldots \vee d_n}$ which is in the image of f.

To see that g is well-defined, for any $x \in X$, $g(x)$ should be a prime filter of $Cl(X)$. Let $V \in Cl(X)$ with $V \in g(x)$ and $V \subseteq W$. Then $x \in W$ and $W \in g(x)$, hence $g(x)$ is an up-set of $Cl(X)$.

If $U, V \in g(x)$, then $x \in U$ and $x \in V$, so $x \in U \cap V$. Thus $U \cap V \in g(x)$ and $g(x)$ is a filter. Suppose $U \cup V \in g(x)$, then $x \in U \cup V$, so $x \in U$ or $x \in V$, hence $g(x)$ is a prime filter.

Let τ_1 be the topology of $PF(Cl(X))$. To check that g is continuous, it suffices to show that the inverse image of every set in the basis of τ_1 is open. If A is an element in the basis of τ_1, since F is a homomorphism,

$$\{ F_U \cap F_V' \mid U, V \in Cl(X) \} = \{ F_{U \cap V'} \mid U, V \in Cl(X) \}$$

So $A = F_W$ for some $W \in Cl(X)$.

$$g^{-1}(A) = \{ x \in X \mid g(x) \in F_W \} = \{ x \in X \mid W \in g(x) \} = W$$

Since W is a clopen subset of X, the inverse image of A is open for every A in the basis of τ_1. Thus, g is continuous.

To show that g is injective, let $x, y \in X$ with $g(x) = g(y)$. Every $a, b \in X$ with $a \notin b$ can be separated by open sets, because X is Hausdorff. But X also has a basis of clopen sets, thus we can separate a, b by clopen sets. Hence,

$$\bigcap g(x) := \bigcap \{ U \in Cl(X) \mid x \in U \} \subseteq \{ x \}$$

If $g(x) = g(y)$, then $\bigcap g(x) = \bigcap g(y)$, hence $\{ x \} = \{ y \}$, which implies $x = y$.

Lastly, we check that g is surjective. Let $P \in PF(Cl(X))$. Since P is a proper filter, it is closed under intersection and does not contain $\varnothing$. It has the finite intersection property and is compact because it is a Stone space. Hence, P has a non-empty intersection. Let $x, y \in P$ with $x \neq y$. By separation, there is a clopen set U with $x \in U$ and $y \in X \setminus U$. Since P is also an ultrafilter, either $U \in P$ or $X \setminus U \in P$. If $U \in P$, then $y \notin \bigcap P$, and if $X \setminus U \in P$, then $x \notin \bigcap P$.

So $\bigcap P = \{ z \}$ for some $z \in X$ and $P \in g(z)$. Since P and $g(x)$ are both prime, maximal filters on $Cl(X)$, $P = g(z)$. Thus, every $P \in PF(Cl(X))$ is in the image of g. □

1.3 Heyting Duality

A Boolean algebra can be generalized to a Heyting algebra by weakening the complement. We say that $a^* := max \{ b \in \mathfrak{L} \mid b \wedge a = 0 \}$ is the pseudo-complement of a. We introduce an implication operation, such that $a^* = a \to 0$.

Definition (Heyting algebra). *A Heyting algebra $\mathfrak{H}$ is a bounded, distributive lattice with a binary implication operation $\to$, such that $c \leq a \to b$ iff $a \wedge c \leq b$.*

Definition (Heyting space). *Let $(X, \tau, \leq)$ be a Stone space with a partial order $\leq$ defined on X. For $x, y \in X$ with $x \nleq y$, if there is a clopen up-set U with $x \in U$ and $y \notin U$, then $(X, \tau, \leq)$ satisfies the Priestley separation axiom. Furthermore, if for every clopen $U \subseteq X$ the set $\downarrow U$ is clopen, then $(X, \tau, \leq)$ is a Heyting space, or an Esakia space.*

Lemma. *If $\mathfrak{H}$ is a Heyting algebra, then $(PF(\mathfrak{H}), \subseteq)$ is a Heyting space.*

Proof. Since $\mathfrak{H}$ is a distributive lattice, $PF(\mathfrak{H})$ is a Stone space and compact. Let $P, Q \in PF(\mathfrak{H})$ with $P \nsubseteq Q$. Then there exists a $x \in P$ with $x \notin Q$, so that $P \in F_x$ but $Q \notin F_x$. We know that F_x is clopen and an up-set of $PF(\mathfrak{H})$. If $R \in F_x$ and $R \subseteq S$, then $x \in S$, so $S \in F_x$. So $PF(\mathfrak{H})$ satisfies the Priestley separation axiom.

We need to verify that if $U \subseteq PF(\mathfrak{H})$ is clopen, then $\downarrow U$ is clopen. Since U is open, it is a union of elements in the basis of the topology of $(PF(\mathfrak{H}), \subseteq)$, which forms an open cover. Since U is closed and $PF(\mathfrak{H})$ is compact, U is compact, so the open cover has a finite subcover. Thus,

$$\downarrow U = \downarrow (\bigcup_{i=1\ldots n} F_{a_i} \cap F'_{b_i}) = \bigcup_{i=1\ldots n} \downarrow (F_{a_i} \cap F'_{b_i})$$

If we can show that $\downarrow (F_{a_i} \cap F_{(b_i)'}) = F_{a\to b}$, then $a \to b \in \mathfrak{H}$, so $F_{a\to b}$ is clopen. Since clopen sets are closed under finite union, we can conclude that $\downarrow U$ is clopen.

Let $a, b \in \mathfrak{H}$, then $a \wedge (a \to b) = a \wedge b \leq b$. Let $P \in F_a \cap F_{a\to b}$. Now, P is a filter and an up-set, so $a \wedge (a \to b) \in P$ and $b \in P$, hence $P \in F_b$. Since $F_a \cap F_{a\to b} \subseteq F_b$, $F_a \cap F'_b \subseteq F'_{a\to b}$. But $F'_{a\to b}$ is a down-set, so that $\downarrow (F_a \cap F'_b) \subseteq F'_{a\to b}$.

Let $P \in F'_{a\to b}$, we want to show that $P \in \downarrow (F_a \cap F'_b)$. Therefore, we need a prime filter Q of $\mathfrak{H}$ with $a \in Q$ but $b \notin Q$, and $P \subseteq Q$. It suffices to show that $a \to b \notin Q$ rather that $b \notin Q$. Such a Q exists if the filter G generated by $P \cup \{a\}$ does not contain $a \to b$. Suppose, towards a contradiction that $a \to b \in G$. Then there exists an $y \in P \cup \{a\}$ with $y \leq a \to b$. Since $a \to b \notin P$, $y = a \wedge x$ for some $x \in P$. Since $a \wedge x \leq a \to b$, $(a \wedge x) \wedge a \leq b$, or $a \wedge x \leq b$, which means that $x \leq a \to b$. But, since $x \in P$, $a \to b \in P$ which is a contradiction!

Hence, $F'_{a\to b} = \downarrow (F_a \cap F'_b)$, and $\downarrow U$ is clopen. □

Lemma. *If $(X, \leq)$ is a Heyting space, then $CU(X, \leq)$ is a Heyting algebra, where $U \to V = (\downarrow (U \cap V'))'$.*

Proof. To see that implication is well-defined, we note that $(\downarrow (U \cap V'))'$ is a clopen up-set for clopen sets U and V. $CU(X, \leq)$ is a bounded distributive lattice by using the properties of the subsets of the powerset. The implication follows the correct properties by a routine application of set-theoretic identities and de Morgan's laws. □

We now establish the isomorphism to complete the Heyting duality.

Theorem (Heyting duality)**.** *The map $f : \mathfrak{H} \to CU(PF(\mathfrak{H}))$ given by $h \mapsto F_h := \{P \in PF(\mathfrak{H}) \mid h \in \mathfrak{H}\}$ is an isomorphism of Heyting algebras. The map $g : (X, \leq) \to PF(CU(X, \leq), \subseteq)$ given by $x \mapsto \{U \in CU(X, \leq) \mid x \in X\}$ is an isomorphism of Heyting spaces.*

Proof. For f to be a homomorphism, we just need to check that it preserves implication. Let $h, k \in \mathfrak{H}$. Then $f(h \to k) = F_{h\to k} = (F'_{h\to k})' = (\downarrow (F_h \cap F'_k))' = F_h \to F_k$.

Injectivity of f follows by the same argument as in the case of Stone duality.

To show that f is surjective, let U be a clopen up-set of $PF(\mathfrak{H})$. Let $P \in U$, $Q \in U'$, then $P \nsubseteq Q$. So there exists some $a_{PQ} \in \mathfrak{H}$ such that $a_{PQ} \in P$ and $a_{PQ} \notin Q$. Hence, $P \in F_{a_{PQ}}$, $Q \in F'_{a_{PQ}}$, and the $F'_{a_{PQ}}$s cover U'. Since $PF(\mathfrak{H})$ is compact,

$$U' \subseteq \bigcup_{i=1...n} (F_{a_{PQ_i}})' = F'_{a_P}$$

where $a_P = a_{PQ_1} \wedge ... \wedge a_{PQ_n}$. Since $P \in F_{a_P Q_i}$ for all a_{PQ_i}, we have $P \in F_{a_P} \subseteq U$.

Since U is the union of the various F_{a_P}, they form an open cover of U. But U is closed and $PF(\mathfrak{H})$ is compact, so U is a finite union

$$\bigcup_{i=1...m} F_{a_{P_i}} = F_a$$

where $a = a_{P_1} \vee ... \vee a_{P_m} \in \mathfrak{H}$. Hence U is in the image of f.

g is well-defined by a similar argument as in the case of Stone duality.

g preserves order because U is an up-set. If $x, y \in X$ with $x \leq y$, $U \in g(x)$ implies $x \in U$, hence $y \in U$, so $U \in g(y)$ and $g(x) \subseteq g(y)$.

Since X and $PF(CU(X, \leq), \subseteq)$ are both Heyting spaces, X is compact and $PF(CU(X, \leq), \subseteq)$ is Hausdorff, so $g(X)$ is closed in $PF(CU(X, \leq), \subseteq)$. Suppose towards a contradiction that g is not surjective, then there is a prime filter P of $CU(X, \leq)$ with $P \notin g(X)$.

So, P and $g(X)$ are closed and disjoint, and there are disjoint open sets U and W with $g(X) \subseteq U$ and $P \in W$. Since W is a union of clopen sets, there is a clopen set V disjoint with $g(X)$ but $P \in V$. Now, V is compact. Let $V = F_S \cap F'_T$ for some $S, T \in CU(X, \leq)$. Then,

$$\emptyset = g^{-1}(V) = g^{-1}(F_S \cap F'_T) = S \cap T'$$

Thus, $S \subseteq T$, which means $V = F_S \cap F_T = \emptyset$, so $P \notin V$ which is a contradiction. Hence, g is surjective.

Finally, we have to check that for every $x \in X$ and $P \in PF(CU(X, \leq), \subseteq)$ with $g(x) \subseteq P$, there is a $y \in X$ with $x \leq y$ and $g(y) = P$. Since g is surjective, there is a $y \in X$ with $P = g(y)$. Since $g(x) \subseteq g(y)$ and g preserves order, we have $x \leq y$. □

As a consequence, if $\mathfrak{H}$ and $\mathfrak{K}$ are isomorphic Heyting algebras, then $PF(\mathfrak{H})$ and $PF(\mathfrak{K})$ are isomorphic Heyting spaces. If X and Y are isomorphic Heyting spaces, then $CU(X)$ and $CU(Y)$ are isomorphic Heyting algebras.

1.4 Completeness of IPL

Using Heyting duality, we get a topological semantics for Intuitionistic Propositional Logic. We can use this to state a completeness theorem for IPL. To do so, we define the Lindenbaum-Tarski algebra by quotienting out the free algebra with the following congruence relation,

$$\varphi \longleftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$$

By using,

$$\| \varphi \| := \{ \psi \mid \vdash \varphi \leftrightarrow \psi \}$$

we define a Heyting algebra $\mathfrak{H}$ as follows.

$$\begin{aligned} 0 &:= \| \bot \| \\ 1 &:= \| \top \| \\ \| \varphi \| \wedge \| \psi \| &:= \| \varphi \wedge \psi \| \\ \| \varphi \| \vee \| \psi \| &:= \| \varphi \vee \psi \| \\ \| \varphi \| \to \| \psi \| &:= \| \varphi \to \psi \| \end{aligned}$$

It is routine to check that this indeed satisfies the axioms of a Heyting algebra. The valuation function is defined as $\mathfrak{v}(\varphi) := \| \varphi \|$. Hence, if $\mathfrak{v}(\varphi) = 1$, we have that $\vdash \varphi \leftrightarrow \top$, and hence $\vdash \varphi$. We say that $\vDash \varphi$ iff $F(\| \varphi \|) = PF(\mathfrak{H})$. By duality, completeness follows.

Theorem (Soundness and Completeness)**.** $\vdash \varphi$ *iff* $\vDash F(\| \varphi \|)$

1.5 Intuitionistic completeness

The proofs of Stone and Heyting duality were carried out in classical set theory, and make use of the *Law of the Excluded Third*, or *Tertium Non Datur (TND)* in several places. Moreover, they rely on the use of the prime ideal theorem and the ultrafilter lemma, which are valid theorems in *ZF set theory with choice.*

From the point of view of categorical logic, these theorems are valid in the internal logic of a boolean, well-pointed topos, where epis split. They will not hold in any general topos, for example, a presheaf topos where choice fails.

It is not possible to prove a strong completeness theorem for Intuitionistic propositional logic using an intuitionistic metatheory, as shown in McCarty et al. [1991].

References

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