Free Commutative Monoids in Homotopy Type Theory

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MFPS XXXVIII, Jul 11, 2022

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Outline

Free commutative monoids

Relational model of Differential Linear Logic

Path space of free commutative monoids

Free commutative monoids

A commutative monoid is a monoid $(M; \cdot, e)$ with a commutation axiom.

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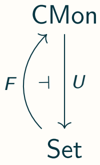

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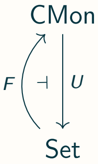

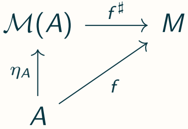

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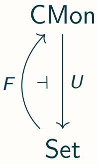

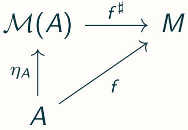

$(\mathcal{M}(A), \eta_A : A \to \mathcal{M}(A))$ is the free commutative monoid on A.

It is characterised by the universal property:

$$(-) \circ \eta_A : \mathsf{CMon}(\mathcal{M}(A), M) \xrightarrow{\sim} (A \to M)$$

Free commutative monoid

How do we *constructively* construct $\mathcal{M}(A)$?

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- unordered lists, or
- lists upto permutation of elements, or
- finite-multisets, or
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We want to define them in univalent type theory:

- without assuming decidable equality,
- and prove the universal property.

Construction of the free commutative monoid

Two easy definitions using HITs:

$$
\begin{aligned}
\underline{\mathsf{ACM}(A)} :\equiv & \\
\eta &: A \to \mathsf{ACM}(A) \\
e &: \mathsf{ACM}(A) \\
- \cdot - &: \mathsf{ACM}(A)^2 \to \mathsf{ACM}(A) \\
\mathsf{assoc} &: x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
\mathsf{unitl} &: e \cdot x = x \\
\mathsf{unitr} &: x \cdot e = x \\
\mathsf{comm} &: x \cdot y = y \cdot x \\
\mathsf{trunc} &: \mathsf{isSet}(\mathsf{ACM}(A))
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\underline{\mathsf{sList}(A)} &:\equiv \\
\mathsf{nil} &: \mathsf{sList}(A) \\
- :: - &: A \times \mathsf{sList}(A) \to \mathsf{sList}(A) \\
\mathsf{swap} &: x :: y :: xs = y :: x :: xs \\
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Both satisfy the categorical universal property of free comm. monoids.

$$\mathcal{M}(A) :\equiv \mathsf{ACM}(A) \simeq_{\mathsf{CMon}} \mathsf{sList}(A)$$

Free commutative monoid monad

- Monad structure:

$$\eta_A : A \to \mathcal{M}(A)$$
$$\mu_A :\equiv (\lambda_{(x\,:\,A)}.\, x)^\sharp : \mathcal{M}(\mathcal{M}(A)) \to \mathcal{M}(A)$$

- Functorial action on $f : A \to B$:

$$\mathcal{M}(f) :\equiv (\lambda_{(a\,:\,A)}.\, \eta_B(fa))^\sharp : \mathcal{M}(A) \to \mathcal{M}(B)$$

- Monad strength:

$$\sigma_{A,B} : \mathcal{M}(A) \times B \to \mathcal{M}(A \times B) : (as, b) \mapsto \mathcal{M}(\lambda_{(a\,:\,A)}.\,(a, b))(as)$$
$$\tau_{A,B} : A \times \mathcal{M}(B) \to \mathcal{M}(A \times B) : (a, bs) \mapsto \mathcal{M}(\lambda_{(b\,:\,B)}.\,(a, b))(bs)$$

- Commutative monad structure:

$$\begin{array}{ccc}
\mathcal{M}(A) \times \mathcal{M}(B) & \xrightarrow{\sigma_{A,\mathcal{M}(B)}} & \mathcal{M}(A \times \mathcal{M}(B)) \\
\Big\downarrow{\scriptstyle \tau_{\mathcal{M}(A),B}} & & \Big\downarrow{\scriptstyle {\tau_{A,B}}^\sharp} \\
\mathcal{M}(\mathcal{M}(A) \times B) & \xrightarrow[{\sigma_{A,B}}^\sharp]{} & \mathcal{M}(A \times B)
\end{array}$$

Free commutative monoid monad

- Strong symmetric monoidal functor:

$$\begin{array}{ccc} \mathcal{M}(A) \times \mathcal{M}(B) & \xrightarrow{\quad\simeq\quad} & \mathcal{M}(A+B) \\ {\scriptstyle \mathcal{M}(\imath_1)\times\mathcal{M}(\imath_2)} \searrow & & \nearrow {\scriptstyle \oplus_{(A+B)}} \\ & \mathcal{M}(A+B) \times \mathcal{M}(A+B) & \end{array}$$

$$\mathbf{1} \xrightarrow[\lambda(x\,:\,\mathbf{1}).\,\varnothing_{\mathbf{0}}]{\simeq} \mathcal{M}(\mathbf{0})$$

- Length function:

$$\ell_A :\equiv \mathcal{M}(\lambda_{(a\,:\,A)}.\,\star) : \mathcal{M}(A) \to \mathcal{M}(\mathbf{1})$$

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Category of relations

Power objects: $\mathfrak{P} : \mathsf{hSet}_i \to \mathsf{hSet}_{i+1} : A \longmapsto (A \to \mathsf{hProp}_i)$.

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Power relative monad:

- unit: $よ_A : A \to \mathfrak{P}(A) : a \longmapsto \lambda_{(x\,:\,A)}.\, a =_A x$
- extension for $f : A \to \mathfrak{P}(B)$:
  $f^* : \mathfrak{P}(A) \to \mathfrak{P}(B) : (\alpha, b) \longmapsto \exists_{(a:A)}.f(a,b) \wedge \alpha(a)$

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Rel has objects hSets and homs $A \nrightarrow B :\equiv A \to \mathfrak{P}(B)$.

- Rel is dagger compact.
- $(-)_* : \mathrm{Set} \to \mathrm{Rel}$ maps functions $f : A \to B$ to relations $よ_B \circ f : A \nrightarrow B$.
- $(-)_*$ preserves coproducts, which become biproducts.

Lifting $\mathcal{M}$ to Rel

$\mathcal{M}$ lifts to the cofree commutative comonoid in Rel.

- comonad structure

$$\delta_A :\equiv ((\mu_A)_*)^\dagger : \mathcal{M}(A) \nrightarrow \mathcal{M}(\mathcal{M}(A))$$

$$\epsilon_A :\equiv ((\eta_A)_*)^\dagger : \mathcal{M}(A) \nrightarrow A$$

- commutative comonoid structure

$$w_A :\equiv ((+\!\!+_A)_*)^\dagger : \mathcal{M}(A) \nrightarrow \mathcal{M}(A) \otimes \mathcal{M}(A)$$

$$k_A :\equiv ((\lambda_{(x\,:\,\mathbf{1})}.\,\mathsf{nil})_*)^\dagger : \mathcal{M}(A) \nrightarrow \mathbf{1}$$

The universal property follows from promonoidal convolution (Day 70).

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- monoidal structure (Seely isomorphisms)

$$\varphi_{A,B} :\equiv (\epsilon_A \otimes \epsilon_B)_\sharp : \mathcal{M}(A) \otimes \mathcal{M}(B) \xrightarrow{\sim} \mathcal{M}(A \otimes B)$$
$$\phi :\equiv (\mathrm{id}_{\mathbf{1}})_\sharp : \mathbf{1} \xrightarrow{\sim} \mathcal{M}(\mathbf{1})$$

Differential Structure

Combinatorics of subsingleton multisets:

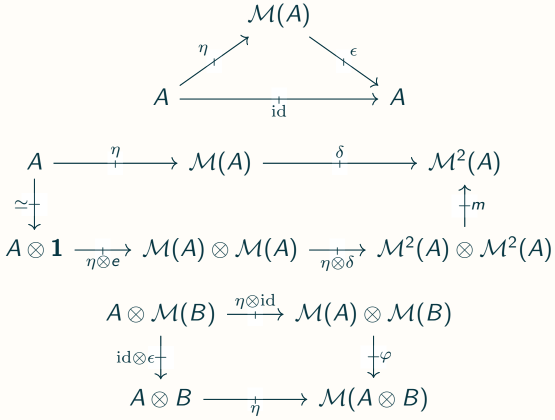
- conical-monoid relation: $as \mathbin{+\!\!+} bs = \mathrm{nil} \iff as = bs = \mathrm{nil}$
- η_A is an embedding: $x =_A y \iff [x] =_{\mathcal{M}(A)} [y]$
- $A \simeq \sum_{as:\mathcal{M}(A)} (\ell(as) = 1) \simeq \sum_{as:\mathcal{M}(A)} \sum_{a:A} (as = [a])$
-
$$[a] = \mu(s) \iff \exists_{(t:\mathcal{M}(\mathcal{M}(A)))}.\ \mu(t) = \mathrm{nil} \wedge [a] :: t = s$$
-
$$[a] = \mathcal{M}(\pi_1)(ps) \wedge bs = \mathcal{M}(\pi_2)(ps) \iff \exists_{(b:B)}.\ bs = [b] \wedge \big[(a, b)\big] = ps$$

Differential Structure

Creation map:

$$\eta_A : A \rightarrow\!\!\!\!\!+ \ \mathcal{M}(A)$$

subject to three laws as follows:



The co-Kleisli category of $\mathcal{M}$:

- has homs $\mathcal{M}(A) \relbar\joinrel\mapsto B$
- is cartesian closed
- is a cartesian differential category

$\mathcal{M}$Rel

The co-Kleisli category of $\mathcal{M}$:

- has homs $\mathcal{M}(A) \twoheadrightarrow B$
- is cartesian closed
- is a cartesian differential category

These are the set-truncated version of generalised species of structures (Fiore, Gambino, Hyland, Winskel 2008).

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Bialgebra law

Every set has a biproduct commutative bialgebra structure.

$$
\begin{array}{ccccc}
A + A & \xrightarrow{\nabla} & A & \xrightarrow{\Delta} & A + A \\
{\scriptstyle \Delta+\Delta}\downarrow & & & & \uparrow{\scriptstyle \nabla+\nabla} \\
A + A + A + A & & \xrightarrow[\mathrm{id}_A + c + \mathrm{id}_A]{} & & A + A + A + A
\end{array}
$$

By the Seely isomorphism, this transfers to the bialgebra law.

$$
\begin{array}{ccccc}
\mathcal{M}(A) \otimes \mathcal{M}(A) & \xrightarrow{m} & \mathcal{M}(A) & \xrightarrow{w} & \mathcal{M}(A) \otimes \mathcal{M}(A) \\
{\scriptstyle w\otimes w}\downarrow & & & & \uparrow{\scriptstyle m\otimes m} \\
\mathcal{M}(A) \otimes \mathcal{M}(A) \otimes \mathcal{M}(A) \otimes \mathcal{M}(A) & & \xrightarrow[\mathrm{id}_{\mathcal{M}(A)} \otimes c \otimes \mathrm{id}_{\mathcal{M}(A)}]{} & & \mathcal{M}(A) \otimes \mathcal{M}(A) \otimes \mathcal{M}(A) \otimes \mathcal{M}(A)
\end{array}
$$

where $c :\equiv (\langle \pi_2, \pi_1 \rangle)_*$ is the symmetry isomorphism.

Commutation relation

Riesz refinement-monoid relation:

$$as \mathbin{+\!\!+} bs = cs \mathbin{+\!\!+} ds$$

$$\iff$$

$$\exists_{(xs_1, xs_2, ys_1, ys_2 : \mathcal{M}(A))}.\ (as = xs_1 \mathbin{+\!\!+} xs_2)\ \wedge\ (bs = ys_1 \mathbin{+\!\!+} ys_2)$$
$$\wedge\ (xs_1 \mathbin{+\!\!+} ys_1 = cs) \wedge (xs_2 \mathbin{+\!\!+} ys_2 = ds)$$

Commutation relation

Riesz refinement-monoid relation:

$$as ++ bs = cs ++ ds$$
$$\iff$$
$$\exists_{(xs_1, xs_2, ys_1, ys_2 : \mathcal{M}(A))}.\ (as = xs_1 ++ xs_2) \ \wedge\ (bs = ys_1 ++ ys_2)$$
$$\wedge\ (xs_1 ++ ys_1 = cs) \wedge (xs_2 ++ ys_2 = ds)$$

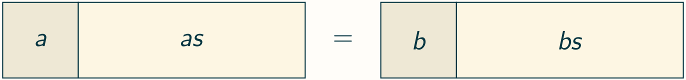
Commutation relation:

$$a :: as = b :: bs$$
$$\Leftrightarrow$$
$$(a = b \wedge as = bs) \vee (\exists_{(cs : \mathcal{M}(A))}.\ as = b :: cs \wedge a :: cs = bs)$$

This commutation relation comes from the creation/annihilation operators associated with the free commutative monoid construction seen as a combinatorial Fock space (Fiore 2015).

Commutation relation

Pointwise equality:



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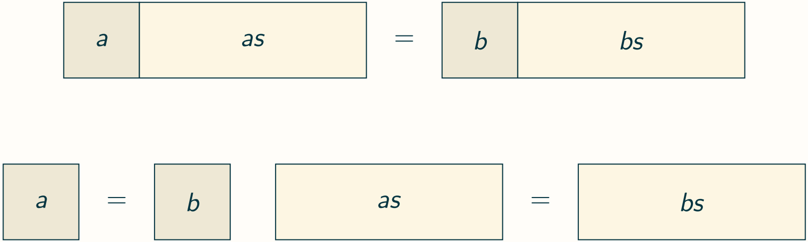

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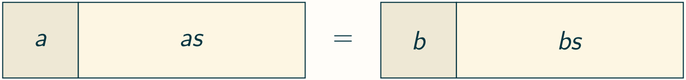
Generalised swapping operation:



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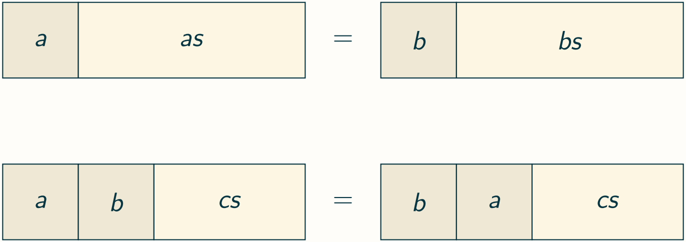

Deduction system

Deduction system for multiset equality:

$$\frac{}{\text{nil} \sim \text{nil}}\ \text{nil-cong} \qquad\qquad \frac{a = b \qquad as \sim bs}{a :: as \sim b :: bs}\ \text{cons-cong}$$

$$\frac{as \sim b :: cs \qquad a :: cs \sim bs}{a :: as \sim b :: bs}\ \text{comm}$$

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The relation $\sim$ generates the path space of $\mathcal{M}(A)$:

$$(as = bs) \Leftrightarrow \|as \sim bs\|\,.$$

Deduction system

The $\sim$ relation is transitive (admits cut):

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Given two deduction trees, we compute the underlying permutations, compose them, and reify it back to a tree (NbE).

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$$\mathsf{vec} \;:\; \mathcal{L}(A) \;\simeq\; \left(\textstyle\sum_{\ell:\mathbb{N}} \mathsf{Fin}_\ell \to A\right) \;:\; \mathsf{list}$$

$$(m, f) \approx_A (n, g) :\equiv (\phi : \mathsf{Fin}_m \xrightarrow{\sim} \mathsf{Fin}_n) \times (f = g \circ \phi) \;.$$

For $as, bs : \mathcal{L}(A)$, we have

$$\mathsf{eval} : \; as \sim_A bs \to \mathsf{vec}(as) \approx_A \mathsf{vec}(bs)$$

and, for $(m, f), (n, g) : \left(\sum_{\ell:\mathbb{N}} \mathsf{Fin}_\ell \to A\right)$, we have

$$\mathsf{quote} : \; (m, f) \approx_A (n, g) \to \mathsf{list}(m, f) \sim_A \mathsf{list}(n, g)$$

Commuted-list construction

The composite $A \to \mathcal{L}(A) \to \mathcal{L}(A)_{/\bar{\sim}_A}$ is the free comm. monoid on A.

Commuted-list construction

The composite $A \to \mathcal{L}(A) \to \mathcal{L}(A)_{/\tilde{\sim}_A}$ is the free comm. monoid on A.

Alternatively, we can define another HIT with a conditional path constructor comm.

$$
\begin{aligned}
\underline{\mathsf{cList}(A)} &:\equiv \\
\mathsf{nil} &: \mathsf{cList}(A) \\
\_ :: \_ &: A \times \mathsf{cList}(A) \to \mathsf{cList}(A) \\
\mathsf{comm} &: \{a\ b : A\}\{as\ bs\ cs : \mathsf{cList}(A)\} \\
&\to (as = b :: cs) \to (a :: cs = bs) \\
&\to a :: as = b :: bs \\
\mathsf{trunc} &: \mathsf{isSet}(\mathsf{cList}(A))
\end{aligned}
$$

Epilogue

Summary:

- Different constructions of free commutative monoids:

$$\mathsf{ACM}(A) \simeq_{\mathsf{CMon}} \mathsf{sList}(A) \simeq_{\mathsf{CMon}} \mathcal{L}(A)_{/\tilde{\approx}_A} \simeq_{\mathsf{CMon}} \mathsf{cList}(A)$$

- Formal construction of the relational model of differential linear logic
- Constructive combinatorics of free commutative monoids:
  - Subsingleton multisets
  - Conical and Refinement-monoid relations
  - Commutation relation
  - Characterisation of the path space
- More details in the paper and formalisation!

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Future work:

- Generalise to free symmetric monoidal groupoids
- Construction of the bicategory of generalised species of structures over groupoids and its differential structure