Weighted Sets & Modalities

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SYCO 8

Motivation

- Programs have intensional information
  - Space Usage
  - Implicit Complexity
  - Effects
- Linear Logic, Modalities
- Linear dependent type theory?

Length Spaces (Hofmann '03)

$$\text{Obj } X = (|X| : \text{Set},\ \ell_X : |X| \to \mathbb{N})$$

$$\text{Hom}(X, Y) = \left\{ f : |X| \to |Y| \;\middle|\; \forall x.\ \ell_Y(f(x)) \leq \ell_X(x) \right\}$$

Cartesian Closed

$$|1| = \{*\} \qquad \ell_1(*) = 0$$

$$|X \times Y| = |X| \times |Y| \qquad \ell_{X \times Y}(x, y) = \max(\ell_X(x), \ell_Y(y))$$

$$|Y^X| = |X| \to |Y| \qquad \ell_{Y^X}(f) = \min \{ n \mid \forall x \in |X|,\ \ell_Y(f(x)) \leq \max(n, \ell_X(x)) \}$$

Symmetric Monoidal Closed

$$I = 1$$

$$|X \otimes Y| = |X| \times |Y| \qquad l_{X \otimes Y}(x, y) = l_X(x) + l_Y(y)$$

$$|X \multimap Y| = |X| \to |Y| \qquad l_{X \multimap Y}(f) = \min\{ n \mid \forall x \in |X|, \; l_Y(f(x)) \leq n + l_X(x) \}$$

Coproducts

$$|X + Y| = |X| + |Y| \qquad l_{X+Y}(i_1(x)) = l_X(x)$$

$$l_{X+Y}(i_2(y)) = l_Y(y)$$

This is a model of Bunched Implication (BI).

But it's not equivalent to $[\mathbb{N}^{op}, \mathrm{Set}]$.

Why does this work?

Families Construction

Let $\mathcal{C}$ be a (small) category.

$\mathrm{Fam}(\mathcal{C})$ has :

Obj $X = \left(|X| : \mathrm{Set},\ w_X : |X| \to \mathrm{Ob}(\mathcal{C}) \right)$

Hom $(X, Y) = \left(f : |X| \to |Y|,\ w_f \right)$

$$\begin{tikzcd} {|X|} \arrow[rr, "f"] \arrow[dr, "w_X"'] & \arrow[d, phantom, "\overset{w_f}{\Rightarrow}"] & {|Y|} \arrow[dl, "w_Y"] \\ & \mathrm{Ob}(\mathcal{C}) & \end{tikzcd}$$

- $\mathrm{Fam}(\mathcal{C})$ has small coproducts

- Every object X is a formal coproduct

$$\sum_{x:|X|} W_X(x)$$

- Universal Property:

Given a category $\mathcal{D}$ with small coproducts and a functor $F:\mathcal{C}\to\mathcal{D}$, there is a unique (up to nat. iso) coproduct preserving functor $F^{\#}:\mathrm{Fam}(\mathcal{C})\to\mathcal{D}$

$$\begin{tikzcd} \mathrm{Fam}(\mathcal{C}) \arrow[r, "F^\#"] & \mathcal{D} \\ \mathcal{C} \arrow[u, "\tilde{y}"] \arrow[ur, "F"'] \arrow[ur, phantom, "\simeq", near start, shift left=3] & \end{tikzcd}$$

$F^\# = \mathrm{Lan}_{\tilde{y}} F$

$PSh(\mathcal{C}) = (\mathrm{Fam}(\mathcal{C}))_{ex}$

$\tilde{y} : \mathcal{C} \longrightarrow \mathrm{Fam}(\mathcal{C})$

$p \longmapsto (\{*\}, * \mapsto p) := 1_p$

- $\mathrm{CoProd}(\mathrm{Fam}(\mathcal{C}), \mathcal{D}) \simeq [\mathcal{C}, \mathcal{D}]$

- $\mathrm{Fam} : \mathrm{Cat} \to \mathrm{CAT}$ is a fully-faithful KZ 2-monad.

Examples

- $\mathrm{Fam}(1) \simeq \mathrm{Set}$
- $\mathrm{Fam}(\mathrm{Set}) \simeq \mathrm{Set}^{\to}$
- $\mathrm{Fam}(\mathrm{Set}^{op})$ is the category of containers / poly.

- If $\mathcal{C}$ has limits, then so does $\mathrm{Fam}(\mathcal{C})$.
- If $\mathcal{C}$ is cartesian closed, then so is $\mathrm{Fam}(\mathcal{C})$
- If $\mathcal{C}$ is symmetric monoidal closed, so is $\mathrm{Fam}(\mathcal{C})$.

(More generally,
- $\mathrm{Fam}(\mathcal{C})$ has limits iff $\mathcal{C}$ has multi-limits.
- If $\mathcal{C}$ has finite products, then $\mathrm{Fam}(\mathcal{C})$ is cartesian-closed iff $\mathcal{C}$ is cartesian multi-closed and has multi-products.)

(How nice are free cocompletions, Adámek & Rosický)

Weighted Sets

Let L be a complete Heyting Algebra (locale).

$$(\leq, 0, 1, \wedge, \vee, [-,-])$$

A weighted set is $X = (|X| : \mathrm{Set},\ w_X : |X| \to L)$.

A weight-preserving map $X \to Y$ is a map $f : |X| \to |Y|$, such that $\forall x.\ w_X(x) \leq w_Y(f(x))$.

$L_+ \,/\, L\text{-Set} \,/\, \mathrm{Fam}(L)$ is the category of weighted sets.

Bi-Cartesian Closed Structure

$$1 = \{*\} \qquad w_1(*) = 1$$

$$|X \times Y| = |X| \times |Y| \qquad w_{X\times Y}(x,y) = w_X(x) \wedge w_Y(y)$$

$$|Y^X| = |X| \to |Y| \qquad w_{Y^X}(f) = \bigwedge_{x:|X|} [w_X(x), w_Y(f(x))]$$

$$|X+Y| = |X| + |Y| \qquad w_{X+Y}(i_1(x)) = w_X(x)$$

$$w_{X+Y}(i_2(y)) = w_Y(y)$$

- Fuzzy Sets, $Fuz(\Omega) / Fam(\Omega) \simeq MonSh(\Omega)$

(Barr '86)

Locally Cartesian-Closed Structure

Pullbacks

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

$$|X \times_A Y| = \{(x,y) \mid f(x) = g(y)\}$$

$$w_{X \times_A Y}(x,y) = w_X(x) \wedge w_Y(y)$$

Families of weighted sets

Let A be a weighted set.

$$(X_a \mid a \in A), \quad w_{X_a} : X_a \to L$$

such that $w_{X_a}(x) \leq w_A(a)$, $\forall x \in X_a$, $a \in A$.

$$X = \coprod (X_a \mid a \in A)$$

$$w_X(x) = w_{X_a}(x), \quad x \in X_a$$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\
& A &
\end{tikzcd}$$

- $f : B \to A$

$$\Delta_f : \mathcal{L}_+/A \longrightarrow \mathcal{L}_+/B$$

$$(X_a \mid a \in A) \mapsto (X_{f(b)} \mid b \in B)$$

$$\Sigma_f : \mathcal{L}_+/B \longrightarrow \mathcal{L}_+/A$$

$$(Y \mid b \in B) \mapsto \left(\sum Y_b \mid a \in A \right)$$

$$(Y_b \mid b \in B) \qquad \Big(\sum_{b\in B_a} \quad \Big|\quad \Big)$$

$$\prod_f : \mathcal{L}_+/B \longrightarrow \mathcal{L}_+/A$$

$$(Y_b \mid b\in B) \longmapsto \Big(\prod_{b\in B_a} \;\Big|\; a\in A\Big)$$

- $\sum_f \dashv \Delta_f \dashv \prod_f$

- If,
$$\begin{tikzcd} B \arrow[r, "v"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "g"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

$$\Delta_g \sum_u \cong \sum_v \Delta_f$$

$$\Delta_g \prod_u \cong \prod_v \Delta_f$$

Modalities

Fix $p \in \mathcal{L}$.

$$\Box_p : \mathcal{L}_+ \longrightarrow \mathcal{L}_+$$

$$(X, w_X) \mapsto \left(\{ x \mid w_X(x) \leq p \}, \; w_X \right)$$

$$\varepsilon_X : \Box_p X \longrightarrow X$$

lex, idempotent

$$\delta_X : \Box_p X \xrightarrow{\sim} \Box_p^2 X$$

Grading

$$\square_p \square_q \cong \square_{p \wedge q} \cong \square_q \square_p$$

$$\square_p \square_1 \cong \square_{p \wedge 1} \cong \square_p$$

Application

$\mathcal{O}(\mathcal{C})$ is complete bi-Heyting.

$$\frac{X \cap A \subseteq B}{X \subseteq A^c \cup B} \qquad \frac{X \cup A \supseteq B}{X \supseteq \underset{\substack{\| \\ B \setminus A}}{A^c \cap B}}$$

- $(\mathcal{O}(\mathcal{C}), \supseteq)$ - weighted sets

$$(X : \mathrm{Set}, \quad w_X : \underset{\substack{\vee \\ \text{track captured variables}}}{X} \to \mathcal{O}(\mathcal{C}))$$

- $f : X \to Y$, $\quad \forall x.\ W_Y(f(x)) \subseteq W_X(x)$.

- $W_1 = \emptyset$

- $W_{A \times B}(a, b) = W_A(a) \cup W_B(b)$

- $W_{B^A}(f) = \bigcup_{a \in A} W_B(f(a)) \setminus W_A(a)$

- $\mathbb{C} = \left(\mathbb{C} : Set, \quad W_{\mathbb{C}}(c) = \{ c \} \right)$

No global sections $1 \to \mathbb{C}$

- $\square X = \{ x \in X \mid W_X(x) = \emptyset \}$

- $|TX| = |X| \times (C \to M)$

  $W_{TX}(x, f) = W_X(x) \cup \{ c \mid f(c) \neq e \}$

- Cancellation : $\phi : \square T X \xrightarrow{\sim} \square X$

Recovering Purity in an Impure Language

λ_c

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x . e : A \to B} \qquad \frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

$$\cdots$$

$$\frac{\Gamma \vdash c : \mathrm{Chan} \quad \Gamma \vdash s : \mathrm{Str}}{\Gamma \vdash c.\mathrm{print}(s) : 1}$$

$$- \quad \frac{\Gamma \vdash e : 1}{\Gamma \vdash e \equiv * : 1} \quad \times$$

Extend with $\Box$

$$\Gamma ::= \cdot \mid \Gamma, x : A^{p} \mid \Gamma, x : A^{i}$$

$$\mathcal{J} ::= \Gamma \vdash e : A \mid \Gamma \vdash^{p} e : A$$

$$\frac{\Gamma^{p} \vdash e : A}{\Gamma \vdash \mathrm{box}(e) : \Box A} \qquad \frac{\Gamma \vdash e_1 : \Box A \quad \Gamma, x : A^{p} \vdash e_2 : B}{\Gamma \vdash \mathrm{let}\ \mathrm{box}(x) = e_1\ \mathrm{in}\ e_2 : B}$$

$$\Gamma^{p} = \{ x : A^{p} \in \Gamma \}$$

(Recovering Purity using Comonads & Capabilities, C. & Krishnaswami)

Interpret in $Kl(T)$

$$\left[\!\left[\frac{\Gamma \vdash c : \mathsf{Chan} \quad \Gamma \vdash s : \mathsf{Str}}{\Gamma \vdash c.\mathsf{print}(s) : 1} \right]\!\right] = \langle c, s \rangle ; \beta ; Tp ; \mu$$

where $p : \mathcal{C} \times \mathcal{S} \to T1$

$$(c, s) \mapsto *, \; \lambda c'. \begin{cases} s & \text{if } c = c' \\ e & \text{o/w} \end{cases}$$

$$\left[\!\left[\frac{\Gamma^{P} \vdash e : A}{\Gamma \vdash \mathsf{box}(e) : \Box A} \right]\!\right] = \rho(\Gamma) ; M(\Gamma) ; \Box \left[\!\left[\Gamma^{P} \vdash e : A \right]\!\right] ; \phi_A ; \eta_{\Box A}$$

Substitution

- If $\Gamma \vdash \theta : \Delta$, $\Delta \vdash e : A$, then $\Gamma \vdash \theta(e) : A$
- $[\![\Gamma \vdash \theta(e) : A]\!] = [\![\Gamma \vdash \theta : \Delta]\!] ; [\![\Delta \vdash e : A]\!]$.

Equational Theory

- If $\Gamma \vdash e_1 \equiv e_2 : A$, then $[\![\Gamma \vdash e_1 : A]\!] = [\![\Gamma \vdash e_2 : A]\!]$

Conservative Extension of λ

$$\underline{A \Rightarrow B} = \underline{\Box A} \Rightarrow \underline{B}$$

$$\underline{\lambda x . e} = \lambda z.\ \mathrm{let}\ \mathrm{box}(x) = z\ \mathrm{in}\ \underline{e}$$

- If $[\![\Gamma \vdash_\lambda e_1 \equiv e_2 : A]\!]$, then $\underline{\Gamma} \vdash \underline{e_1} \equiv \underline{e_2} : \underline{A}$

- If $\Gamma \vdash_\lambda e_1 : A$, $\Gamma \vdash_\lambda e_2 : A$, and $\underline{\Gamma} \vdash \underline{e_1} \equiv \underline{e_2} : \underline{A}$,

  then $\Gamma \vdash_\lambda e_1 \equiv e_2 : A$.

Concluding Thoughts

- Other examples?
- Linear dependent types?
- Interaction of monads / comonads?